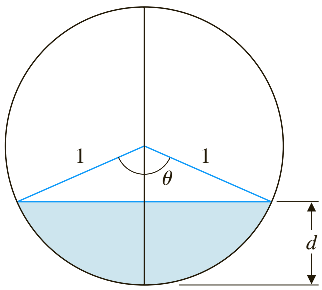
To simplify calculations, suppose the circle is a unit circle with center at (0, 0). Sketch radii extending from the origin to the top of the oil. The area of oil at the bottom equals the area of the portion of the circle bounded by the radii minus the area of the triangle formed above the oil in the figure.



Start with the triangle, which has area one-half base times height. Explain why the height is $1 - d$. Find a right triangle in the figure (there are two of them) with hypotenuse 1 (the radius of the circle) and one vertical side of length $1 - d$. The horizontal side has length equal to one-half the base of the larger triangle. Show that this equals $\sqrt{1-(1-d)^2}$. The area of the portion of the circle equals $\pi\theta/2\pi = \theta/2$, where θ is the angle at the top of the triangle. Find this angle as a function of d. (Hint: Go back to the right triangle used above with upper angle $\theta/2$.) Then find the area filled with oil and divide by π to get the portion of the tank filled with oil.

3. Computer graphics can be misleading. This exercise works best using a "disconnected" graph (individual dots, not connected). Graph $y = \sin x^2$ using a graphing window for which each pixel represents a step of 0.1 in the x- or y-direction. You should get the impression of a sine wave that oscillates more and more rapidly as you move to the left and right. Next, change the graphing window so that the middle of the original screen (probably $x = 0$) is at the far left of the new screen. You will likely see what appears to be a random jumble of dots. Continue to change the graphing window by increasing the x-values. Describe the patterns or lack of patterns that you see. You should find one pattern that looks like two rows of dots across the top and bottom of the screen; another pattern looks like the original sine wave. For each pattern that you find, pick adjacent points with x-coordinates a and b. Then change the graphing window so that $a \le x \le b$ and find the portion of the graph that is missing. Remember that, whether the points are connected or not, computer graphs always leave out part of the graph; it is part of your job to know whether or not the missing part is important.

0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Some bacteria reproduce very quickly, as you may have discovered if you have ever had an infected cut or strep throat. Under the right circumstances, the number of bacteria in certain cultures will double in as little as an hour. In this section, we discuss some functions that can be used to model such rapid growth.

Suppose that initially there are 100 bacteria at a given site and the population doubles every hour. Call the population function $P(t)$, where t represents time (in hours) and start the clock running at time $t = 0$. Since the initial population is 100, we have $P(0) = 100$. After 1 hour, the population will double to 200, so that $P(1) = 200$. After another hour, the population will have doubled again to 400, making $P(2) = 400$ and so on.

To compute the bacterial population after 10 hours, you could calculate the population at 4 hours, 5 hours and so on, or you could use the following shortcut. To find $P(1)$, double the initial population, so that $P(1) = 2 \cdot 100$. To find $P(2)$, double the population at time $t = 1$, so that $P(2) = 2 \cdot 2 \cdot 100 = 2^2 \cdot 100$. Similarly, $P(3) = 2^3 \cdot 100$. This pattern leads us to

$$P(10) = 2^{10} \cdot 100 = 102{,}400.$$

Observe that the population can be modeled by the function

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an **exponential** function because the variable t is in the exponent. There is a subtle question here: what is the domain of this function? We have so far used only integer

values of t, but for what other values of t does $P(t)$ make sense? Certainly, rational powers make sense, as in $P(1/2) = 2^{1/2} \cdot 100$, where $2^{1/2} = \sqrt{2}$. This says that the number of bacteria in the culture after a half hour is approximately

$$P(1/2) = 2^{1/2} \cdot 100 = \sqrt{2} \cdot 100 \approx 141.$$

It's a simple matter to interpret fractional powers as roots. For instance,

$$\begin{aligned} x^{1/2} &= \sqrt{x}, \\ x^{1/3} &= \sqrt[3]{x}, \\ x^{1/4} &= \sqrt[4]{x}, \\ x^{2/3} &= \sqrt[3]{x^2} = (\sqrt[3]{x})^2, \\ x^{3.1} &= x^{31/10} = \sqrt[10]{x^{31}} \end{aligned}$$

and so on. But, what about irrational powers? They are harder to define, but they work exactly the way you would want them to. For instance, since π is between 3.14 and 3.15, 2^π is between $2^{3.14}$ and $2^{3.15}$. In this way, we define 2^x for x irrational to fill in the gaps in the graph of $y = 2^x$ for x rational. That is, if x is irrational and $a < x < b$, for rational numbers a and b, then $2^a < 2^x < 2^b$. This is the logic behind the definition of irrational powers.

If for some reason you wanted to find the bacterial population after π hours, you can use your calculator or computer to obtain the approximate population:

$$P(\pi) = 2^\pi \cdot 100 \approx 882.$$

For your convenience, we now summarize the usual rules of exponents.

RULES OF EXPONENTS

- For any integers m and n,
$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m.$$
- For any real number p,
$$x^{-p} = \frac{1}{x^p}.$$
- For any real numbers p and q,
$$(x^p)^q = x^{p \cdot q}.$$
- For any real numbers p and q,
$$x^p \cdot x^q = x^{p+q}.$$

Throughout your calculus course, you will need to be able to quickly convert back and forth between exponential form and fractional or root form.

EXAMPLE 5.1 Converting Expressions to Exponential Form

Convert each to exponential form: (a) $3\sqrt{x^5}$, (b) $\dfrac{5}{\sqrt[3]{x}}$, (c) $\dfrac{3x^2}{2\sqrt{x}}$ and (d) $(2^x \cdot 2^{3+x})^2$.

Solution For (a), simply leave the 3 alone and convert the power:

$$3\sqrt{x^5} = 3x^{5/2}.$$

For (b), use a negative exponent to write x in the numerator:

$$\frac{5}{\sqrt[3]{x}} = 5x^{-1/3}.$$

For (c), first separate the constants from the variables and then simplify:

$$\frac{3x^2}{2\sqrt{x}} = \frac{3}{2}\frac{x^2}{x^{1/2}} = \frac{3}{2}x^{2-1/2} = \frac{3}{2}x^{3/2}.$$

For (d), first work inside the parentheses and then square:

$$(2^x \cdot 2^{3+x})^2 = (2^{x+3+x})^2 = (2^{2x+3})^2 = 2^{4x+6}.$$

The function in part (d) of example 5.1 is called an *exponential* function with a *base* of 2.

DEFINITION 5.1

For any constant $b > 0$, the function $f(x) = b^x$ is called an **exponential function.** Here, b is called the **base** and x is the **exponent.**

Be careful to distinguish between algebraic functions such as $f(x) = x^3$ and $g(x) = x^{2/3}$ and exponential functions. For exponential functions such as $h(x) = 2^x$, the variable is in the exponent (hence the name), instead of in the base. Also, notice that the domain of an exponential function is the entire real line, $(-\infty, \infty)$, while the range is the open interval $(0, \infty)$, since $b^x > 0$ for all x.

While any positive real number can be used as a base for an exponential function, three bases are the most commonly used in practice. Base 2 arises naturally when analyzing processes that double at regular intervals (such as the bacteria at the beginning of this section). Our standard counting system is base 10, so this base is commonly used. However, far and away the most useful base is the irrational number e. Like π, the number e has a surprising tendency to occur in important calculations. We define e by

$$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.} \tag{5.1}$$

Note that equation (5.1) has at least two serious shortcomings. First, we have not yet said what the notation $\lim\limits_{n\to\infty}$ means. (In fact, we won't define this until Chapter 1.) Second, it's unclear why anyone would ever define a number in such a strange way. We will not be in a position to answer the second question until Chapter 4 (but the answer is worth the wait).

It suffices for the moment to say that equation (5.1) means that e can be approximated by calculating values of $(1 + 1/n)^n$ for large values of n and that the larger the value of n, the closer the approximation will be to the actual value of e. In particular, if you look at the sequence of numbers $(1 + 1/2)^2$, $(1 + 1/3)^3$, $(1 + 1/4)^4$ and so on, they will get progressively closer and closer to (i.e., home in on) the irrational number e.

To get an idea of the value of e, compute several of these numbers:

$$\left(1+\frac{1}{10}\right)^{10} = 2.5937\ldots,$$

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.7169\ldots,$$

$$\left(1+\frac{1}{10{,}000}\right)^{10{,}000} = 2.7181\ldots$$

and so on. You should compute enough of these values to convince yourself that the first few digits of the decimal representation of e ($e \approx 2.718281828459\ldots$) are correct.

EXAMPLE 5.2 Computing Values of Exponentials

Approximate e^4, $e^{-1/5}$ and e^0.

Solution From a calculator, we find that

$$e^4 = e \cdot e \cdot e \cdot e \approx 54.598.$$

From the usual rules of exponents,

$$e^{-1/5} = \frac{1}{e^{1/5}} = \frac{1}{\sqrt[5]{e}} \approx 0.81873.$$

(On a calculator, it is convenient to replace $-1/5$ with -0.2.) Finally, $e^0 = 1$. ■

The graphs of the exponential functions summarize many of their important properties.

EXAMPLE 5.3 Sketching Graphs of Exponentials

Sketch the graphs of the exponential functions $y = 2^x$, $y = e^x$, $y = e^{2x}$, $y = e^{x/2}$, $y = (1/2)^x$ and $y = e^{-x}$.

Solution Using a calculator or computer, you should get graphs similar to those that follow.

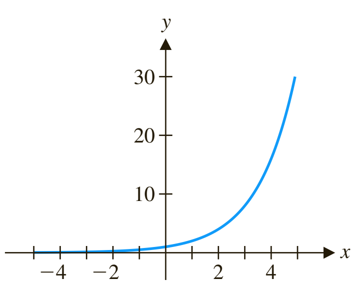


FIGURE 0.69a
$y = 2^x$

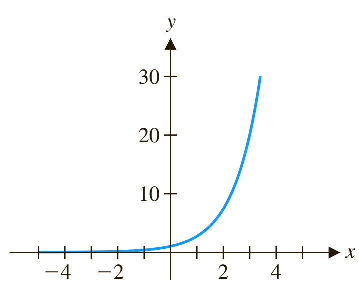


FIGURE 0.69b
$y = e^x$

Notice that each of the graphs in Figures 0.69a, 0.69b, 0.70a and 0.70b starts very near the x-axis (reading left to right), passes through the point (0, 1) and then rises steeply. This is true for all exponentials with base greater than 1 and with a positive coefficient in the exponent. Note that the larger the base ($e > 2$) or the larger the coefficient in the

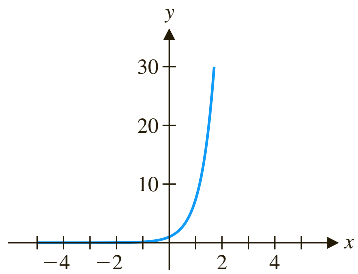


FIGURE 0.70a
$y = e^{2x}$

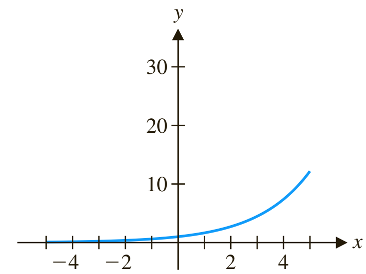


FIGURE 0.70b
$y = e^{x/2}$

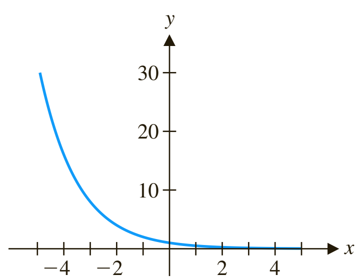


FIGURE 0.71a
$y = (1/2)^x$

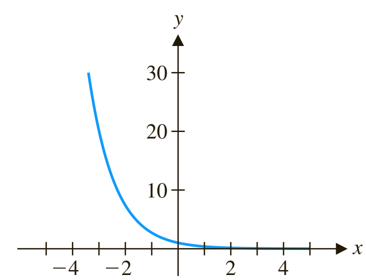


FIGURE 0.71b
$y = e^{-x}$

exponent $(2 > 1 > 1/2)$, the more quickly the graph rises to the right (and drops to the left). Note that the graphs in Figures 0.71a and 0.71b are the mirror images in the y-axis of Figures 0.69a and 0.69b, respectively. The graphs rise as you move to the left and drop toward the x-axis as you move to the right. It's worth noting that by the rules of exponents, $(1/2)^x = 2^{-x}$ and $(1/e)^x = e^{-x}$. ■

In Figures 0.69–0.71, each exponential function is one-to-one and, hence, has an inverse function. We define the logarithmic functions to be inverses of the exponential functions.

DEFINITION 5.2

For any positive number $b \neq 1$, the **logarithm** function with base b, written $\log_b x$, is defined by

$$y = \log_b x \quad \text{if and only if} \quad x = b^y.$$

That is, the logarithm $\log_b x$ gives the exponent to which you must raise the base b to get the given number x. For example,

$$\begin{aligned} \log_{10} 10 &= 1 \quad (\text{since } 10^1 = 10), \\ \log_{10} 100 &= 2 \quad (\text{since } 10^2 = 100), \\ \log_{10} 1000 &= 3 \quad (\text{since } 10^3 = 1000) \end{aligned}$$

and so on. The value of $\log_{10} 45$ is less clear than the preceding three values, but the idea is the same: you need to find the number y such that $10^y = 45$. The answer is between

1 and 2, but to be more precise, you will need to employ trial and error. You should get $\log_{10} 45 \approx 1.6532$.

Observe from Definition 5.2 that for any base $b > 0\,(b \neq 1)$, if $y = \log_b x$, then $x = b^y > 0$. That is, the domain of $f(x) = \log_b x$ is the interval $(0, \infty)$. Likewise, the range of f is the entire real line, $(-\infty, \infty)$.

As with exponential functions, the most useful bases turn out to be 2, 10, and e. We usually abbreviate $\log_{10} x$ by $\log x$. Similarly, $\log_e x$ is usually abbreviated $\ln x$ (for **natural logarithm**).

EXAMPLE 5.4 Evaluating Logarithms

Without using your calculator, determine $\log(1/10)$, $\log(0.001)$, $\ln e$ and $\ln e^3$.

Solution Since $1/10 = 10^{-1}$, $\log(1/10) = -1$. Similarly, since $0.001 = 10^{-3}$, we have that $\log(0.001) = -3$. Since $\ln e = \log_e e^1$, $\ln e = 1$. Similarly, $\ln e^3 = 3$. ■

We want to emphasize the inverse relationship defined by Definition 5.2. That is, b^x and $\log_b x$ are inverse functions for any $b > 0\,(b \neq 1)$.

In particular, for the base e, we have

$$\boxed{e^{\ln x} = x \quad \text{for any } x > 0 \qquad \text{and} \qquad \ln(e^x) = x \quad \text{for any } x.} \tag{5.2}$$

We demonstrate this as follows. Let

$$y = \ln x = \log_e x.$$

By Definition 5.2, we have that

$$x = e^y = e^{\ln x}.$$

We can use this relationship between natural logarithms and exponentials to solve equations involving logarithms and exponentials, as in examples 5.5 and 5.6.

EXAMPLE 5.5 Solving a Logarithmic Equation

Solve the equation $\ln(x + 5) = 3$ for x.

Solution Taking the exponential of both sides of the equation and writing things backward (for convenience), we have

$$e^3 = e^{\ln(x+5)} = x + 5,$$

from (5.2). Subtracting 5 from both sides gives us

$$e^3 - 5 = x. \ ■$$

EXAMPLE 5.6 Solving an Exponential Equation

Solve the equation $e^{x+4} = 7$ for x.

Solution Taking the natural logarithm of both sides and writing things backward (for simplicity), we have from (5.2) that

$$\ln 7 = \ln(e^{x+4}) = x + 4.$$

Subtracting 4 from both sides yields

$$\ln 7 - 4 = x. \ ■$$

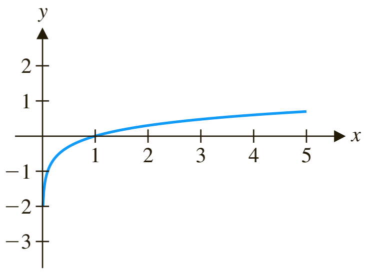


FIGURE 0.72a
$y = \log x$

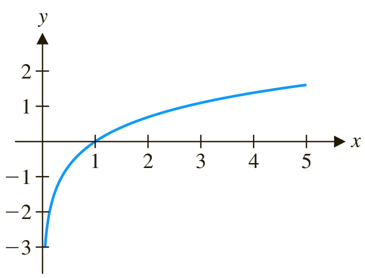


FIGURE 0.72b
$y = \ln x$

As always, graphs provide excellent visual summaries of the important properties of a function.

EXAMPLE 5.7 Sketching Graphs of Logarithms

Sketch graphs of $y = \log x$ and $y = \ln x$, and briefly discuss the properties of each.

Solution From a calculator or computer, you should obtain graphs resembling those in Figures 0.72a and 0.72b. Notice that both graphs appear to have a vertical asymptote at $x = 0$ (why would that be?), cross the x-axis at $x = 1$ and very gradually increase as x increases. Neither graph has any points to the left of the y-axis, since $\log x$ and $\ln x$ are defined only for $x > 0$. The two graphs are very similar, although not identical. ■

The properties just described graphically are summarized in Theorem 5.1.

THEOREM 5.1

For any positive base $b \neq 1$,

(i) $\log_b x$ is defined only for $x > 0$,
(ii) $\log_b 1 = 0$ and
(iii) if $b > 1$, then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

PROOF

(i) Note that since $b > 0$, $b^y > 0$ for any y. So, if $\log_b x = y$, then $x = b^y > 0$.
(ii) Since $b^0 = 1$ for any number $b \neq 0$, $\log_b 1 = 0$ (i.e., the exponent to which you raise the base b to get the number 1 is 0).
(iii) We leave this as an exercise. ■

All logarithms share a set of defining properties, as stated in Theorem 5.2.

THEOREM 5.2

For any positive base $b \neq 1$ and positive numbers x and y, we have

(i) $\log_b(xy) = \log_b x + \log_b y$,
(ii) $\log_b(x/y) = \log_b x - \log_b y$ and
(iii) $\log_b(x^y) = y \log_b x$.

As with most algebraic rules, each one of these properties can dramatically simplify calculations when it applies.

EXAMPLE 5.8 Simplifying Logarithmic Expressions

Write each as a single logarithm: (a) $\log_2 27^x - \log_2 3^x$ and (b) $\ln 8 - 3\ln(1/2)$.

Solution First, note that there is more than one order in which to work each problem. For part (a), we have $27 = 3^3$ and so, $27^x = (3^3)^x = 3^{3x}$. This gives us

$$\begin{aligned}\log_2 27^x - \log_2 3^x &= \log_2 3^{3x} - \log_2 3^x \\ &= 3x \log_2 3 - x \log_2 3 = 2x \log_2 3 = \log_2 3^{2x}.\end{aligned}$$

For part (b), note that $8 = 2^3$ and $1/2 = 2^{-1}$. Then,

$$\begin{aligned}\ln 8 - 3\ln(1/2) &= 3\ln 2 - 3(-\ln 2)\\ &= 3\ln 2 + 3\ln 2 = 6\ln 2 = \ln 2^6 = \ln 64.\end{aligned}$$

In some circumstances, it is beneficial to use the rules of logarithms to expand a given expression, as in example 5.9.

EXAMPLE 5.9 Expanding a Logarithmic Expression

Use the rules of logarithms to expand the expression $\ln\left(\frac{x^3y^4}{z^5}\right)$.

Solution From Theorem 5.2, we have that

$$\begin{aligned}\ln\left(\frac{x^3y^4}{z^5}\right) &= \ln(x^3y^4) - \ln(z^5) = \ln(x^3) + \ln(y^4) - \ln(z^5)\\ &= 3\ln x + 4\ln y - 5\ln z.\end{aligned}$$

Using the rules of exponents and logarithms, we can rewrite any exponential as an exponential with base e, as follows. For any base $a > 0$, we have

$$\boxed{a^x = e^{\ln(a^x)} = e^{x\ln a}.} \tag{5.3}$$

This follows from Theorem 5.2 (iii) and the fact that $e^{\ln y} = y$, for all $y > 0$.

EXAMPLE 5.10 Rewriting Exponentials as Exponentials with Base e

Rewrite the exponentials 2^x, 5^x and $(2/5)^x$ as exponentials with base e.

Solution From (5.3), we have

$$2^x = e^{\ln(2^x)} = e^{x\ln 2},$$

$$5^x = e^{\ln(5^x)} = e^{x\ln 5}$$

and

$$\left(\frac{2}{5}\right)^x = e^{\ln[(2/5)^x]} = e^{x\ln(2/5)}.$$

Just as we can rewrite an exponential with any positive base in terms of an exponential with base e, we can rewrite any logarithm in terms of natural logarithms, as follows. For any positive base $b\,(b \neq 1)$, we will show that

$$\boxed{\log_b x = \frac{\ln x}{\ln b}.} \tag{5.4}$$

Let $y = \log_b x$. Then by Definition 5.2, we have that $x = b^y$. Taking the natural logarithm of both sides of this equation, we get by Theorem 5.2 (iii) that

$$\ln x = \ln(b^y) = y\ln b.$$

Dividing both sides by $\ln b$ (since $b \neq 1$, $\ln b \neq 0$) gives us

$$y = \frac{\ln x}{\ln b},$$

establishing (5.4).

Equation (5.4) is useful for computing logarithms with bases other than e or 10. This is important since, more than likely, your calculator has keys only for $\ln x$ and $\log x$. We illustrate this idea in example 5.11.

EXAMPLE 5.11 Approximating the Value of Logarithms

Approximate the value of $\log_7 12$.

Solution From (5.4), we have

$$\log_7 12 = \frac{\ln 12}{\ln 7} \approx 1.2769894.$$

Saint Louis Gateway Arch

Hyperbolic Functions

There are two special combinations of exponential functions, called the **hyperbolic sine** and **hyperbolic cosine** functions, that have important applications. For instance, the Gateway Arch in Saint Louis was built in the shape of a hyperbolic cosine graph. (See the photograph in the margin.) The hyperbolic sine function [denoted by $\sinh(x)$] and the hyperbolic cosine function [denoted by $\cosh(x)$] are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Graphs of these functions are shown in Figures 0.73a and 0.73b. The hyperbolic functions (including the hyperbolic tangent, $\tanh x$, defined in the expected way) are often convenient to use when solving equations. For now, we verify several basic properties that the hyperbolic functions satisfy in parallel with their trigonometric counterparts.

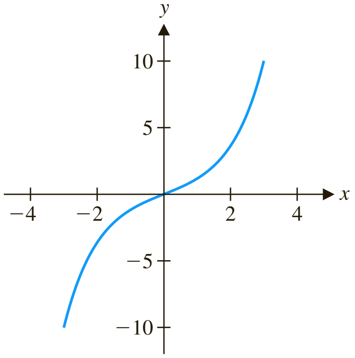


FIGURE 0.73a
$y = \sinh x$

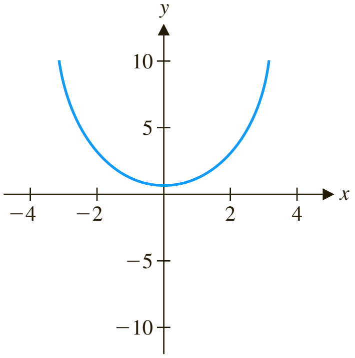


FIGURE 0.73b
$y = \cosh x$

EXAMPLE 5.12 Computing Values of Hyperbolic Functions

Compute $f(0)$, $f(1)$ and $f(-1)$, and determine how $f(x)$ and $f(-x)$ compare for each function: (a) $f(x) = \sinh x$ and (b) $f(x) = \cosh x$.

Solution For part (a), we have $\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$. Note that this means that $\sinh 0 = \sin 0 = 0$. Also, we have $\sinh 1 = \frac{e^1 - e^{-1}}{2} \approx 1.18$, while $\sinh(-1) = \frac{e^{-1} - e^1}{2} \approx -1.18$. Notice that $\sinh(-1) = -\sinh 1$. In fact, for any x,

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh x.$$

[The same rule holds for the sine function: $\sin(-x) = -\sin x$.] For part (b), we have $\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$. Note that this means that $\cosh 0 = \cos 0 = 1$. Also, we have $\cosh 1 = \frac{e^1 + e^{-1}}{2} \approx 1.54$, while $\cosh(-1) = \frac{e^{-1} + e^1}{2} \approx 1.54$. Notice that $\cosh(-1) = \cosh 1$. In fact, for any x,

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

[The same rule holds for the cosine function: $\cos(-x) = \cos x$.] ■

○ Fitting a Curve to Data

You are familiar with the idea that two points determine a straight line. As we see in example 5.13, two points will also determine an exponential function.

EXAMPLE 5.13 Matching Data to an Exponential Curve

Find the exponential function of the form $f(x) = ae^{bx}$ that passes through the points (0, 5) and (3, 9).

Solution We must solve for a and b, using the properties of logarithms and exponentials. First, for the graph to pass through the point (0, 5), this means that

$$5 = f(0) = ae^{b \cdot 0} = a,$$

so that $a = 5$. Next, for the graph to pass through the point (3, 9), we must have

$$9 = f(3) = ae^{3b} = 5e^{3b}.$$

To solve for b, we divide both sides of the equation by 5 and take the natural logarithm of both sides, which yields

$$\ln\left(\frac{9}{5}\right) = \ln e^{3b} = 3b,$$

from (5.2). Finally, dividing by 3 gives us the value for b:

$$b = \frac{1}{3}\ln\left(\frac{9}{5}\right).$$

Thus, $f(x) = 5e^{\frac{1}{3}\ln(9/5)x}$. ■

Year	*U.S. Population*
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,866,020
1840	17,069,453
1850	23,191,876
1860	31,443,321

Consider the population of the United States from 1790 to 1860, found in the accompanying table. A plot of these data points can be seen in Figure 0.74 (where the vertical scale represents the population in millions). This shows that the population was increasing, with larger and larger increases each decade. If you sketch an imaginary curve through these points, you will probably get the impression of a parabola or perhaps the right half of a

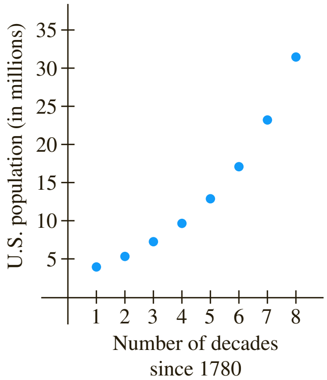


FIGURE 0.74
U.S. Population 1790–1860

cubic or exponential. And that's the question: are these data best modeled by a quadratic function, a cubic function, an exponential function or what?

We can use the properties of logarithms from Theorem 5.2 to help determine whether a given set of data is modeled better by a polynomial or an exponential function, as follows. Suppose that the data actually come from an exponential, say, $y = ae^{bx}$ (i.e., the data points lie on the graph of this exponential). Then,

$$\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx} = \ln a + bx.$$

If you draw a new graph, where the horizontal axis shows values of x and the vertical axis corresponds to values of $\ln y$, then the graph will be the line $\ln y = bx + c$ (where the constant $c = \ln a$). On the other hand, suppose the data actually came from a polynomial. If $y = bx^n$ (for any n), then observe that

$$\ln y = \ln(bx^n) = \ln b + \ln x^n = \ln b + n \ln x.$$

In this case, a graph with horizontal and vertical axes corresponding to x and $\ln y$, respectively, will look like the graph of a logarithm, $\ln y = n \ln x + c$. Such **semi-log graphs** (i.e., graphs of $\ln y$ versus x) let us distinguish the graph of an exponential from that of a polynomial: graphs of exponentials become straight lines, while graphs of polynomials (of degree ≥ 1) become logarithmic curves. Scientists and engineers frequently use semi-log graphs to help them gain an understanding of physical phenomena represented by some collection of data.

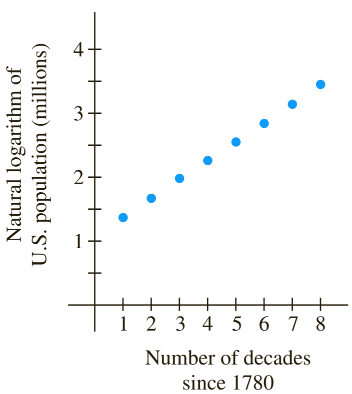


FIGURE 0.75
Semi-log plot of U.S. population

EXAMPLE 5.14 Using a Semi-Log Graph to Identify a Type of Function

Determine whether the population of the United States from 1790 to 1860 was increasing exponentially or as a polynomial.

Solution As already indicated, the trick is to draw a semi-log graph. That is, instead of plotting (1, 3.9) as the first data point, plot (1, ln 3.9) and so on. A semi-log plot of this data set is seen in Figure 0.75. Although the points are not exactly colinear (how would you prove this?), the plot is very close to a straight line with $\ln y$-intercept of 1 and slope 0.3. You should conclude that the population is well modeled by an exponential function. The exponential model would be $y = P(t) = ae^{bt}$, where t represents the number of decades since 1780. Here, b is the slope and $\ln a$ is the $\ln y$-intercept of the line in the semi-log graph. That is, $b \approx 0.3$ and $\ln a \approx 1$ (why?), so that $a \approx e$. The population is then modeled by

$$P(t) = e \cdot e^{0.3t} \text{ million.}$$ ■

EXERCISES 0.5

WRITING EXERCISES

1. Starting from a single cell, a human being is formed by 50 generations of cell division. Explain why after n divisions there are 2^n cells. Guess how many cells will be present after 50 divisions, then compute 2^{50}. Briefly discuss how rapidly exponential functions increase.

2. Explain why the graphs of $f(x) = 2^{-x}$ and $g(x) = \left(\frac{1}{2}\right)^x$ are the same.

3. Compare $f(x) = x^2$ and $g(x) = 2^x$ for $x = \frac{1}{2}, x = 1$, $x = 2$, $x = 3$ and $x = 4$. In general, which function is bigger for large values of x? For small values of x?

4. Compare $f(x) = 2^x$ and $g(x) = 3^x$ for $x = -2, x = -\frac{1}{2}$, $x = \frac{1}{2}$ and $x = 2$. In general, which function is bigger for negative values of x? For positive values of x?

In exercises 1–6, convert each exponential expression into fractional or root form.

1. 2^{-3} **2.** 4^{-2} **3.** $3^{1/2}$

4. $6^{2/5}$ **5.** $5^{2/3}$ **6.** $4^{-2/3}$

In exercises 7–12, convert each expression into exponential form.

7. $\dfrac{1}{x^2}$ **8.** $\sqrt[3]{x^2}$ **9.** $\dfrac{2}{x^3}$

10. $\dfrac{4}{x^2}$ **11.** $\dfrac{1}{2\sqrt{x}}$ **12.** $\dfrac{3}{2\sqrt{x^3}}$

In exercises 13–16, find the integer value of the given expression without using a calculator.

13. $4^{3/2}$ **14.** $8^{2/3}$ **15.** $\dfrac{\sqrt{8}}{2^{1/2}}$ **16.** $\dfrac{2}{(1/3)^2}$

In exercises 17–20, use a calculator or computer to estimate each value.

17. $2e^{-1/2}$ **18.** $4e^{-2/3}$

19. $\dfrac{12}{e}$ **20.** $\dfrac{14}{\sqrt{e}}$

In exercises 21–30, sketch a graph of the given function.

21. $f(x) = e^{2x}$ **22.** $f(x) = e^{3x}$

23. $f(x) = 2e^{x/4}$ **24.** $f(x) = e^{-x^2}$

25. $f(x) = 3e^{-2x}$ **26.** $f(x) = 10e^{-x/3}$

27. $f(x) = \ln 2x$ **28.** $f(x) = \ln x^2$

29. $f(x) = e^{2\ln x}$ **30.** $f(x) = e^{-x/4} \sin x$

In exercises 31–40, solve the given equation for x.

31. $e^{2x} = 2$ **32.** $e^{4x} = 3$

33. $e^x(x^2 - 1) = 0$ **34.** $xe^{-2x} + 2e^{-2x} = 0$

35. $\ln 2x = 4$ **36.** $2\ln 3x = 1$

37. $4\ln x = -8$ **38.** $x^2 \ln x - 9\ln x = 0$

39. $e^{2\ln x} = 4$ **40.** $\ln(e^{2x}) = 6$

In exercises 41 and 42, use the definition of logarithm to determine the value.

41. (a) $\log_3 9$ (b) $\log_4 64$ (c) $\log_3 \frac{1}{27}$

42. (a) $\log_4 \frac{1}{16}$ (b) $\log_4 2$ (c) $\log_9 3$

In exercises 43 and 44, use equation (5.4) to approximate the value.

43. (a) $\log_3 7$ (b) $\log_4 60$ (c) $\log_3 \frac{1}{24}$

44. (a) $\log_4 \frac{1}{10}$ (b) $\log_4 3$ (c) $\log_9 8$

In exercises 45–50, rewrite the expression as a single logarithm.

45. $\ln 3 - \ln 4$ **46.** $2\ln 4 - \ln 3$

47. $\frac{1}{2}\ln 4 - \ln 2$ **48.** $3\ln 2 - \ln \frac{1}{2}$

49. $\ln \frac{3}{4} + 4\ln 2$ **50.** $\ln 9 - 2\ln 3$

In exercises 51–54, find a function of the form $f(x) = ae^{bx}$ with the given function values.

51. $f(0) = 2,\ f(2) = 6$ **52.** $f(0) = 3,\ f(3) = 4$

53. $f(0) = 4,\ f(2) = 2$ **54.** $f(0) = 5,\ f(1) = 2$

55. A fast-food restaurant gives every customer a game ticket. With each ticket, the customer has a 1-in-10 chance of winning a free meal. If you go 10 times, estimate your chances of winning at least one free meal. The exact probability is $1 - \left(\frac{9}{10}\right)^{10}$. Compute this number and compare it to your guess.

56. In exercise 55, if you had 20 tickets with a 1-in-20 chance of winning, would you expect your probability of winning at least once to increase or decrease? Compute the probability $1 - \left(\frac{19}{20}\right)^{20}$ to find out.

57. In general, if you have n chances of winning with a 1-in-n chance on each try, the probability of winning at least once is $1 - \left(1 - \frac{1}{n}\right)^n$. As n gets larger, what number does this probability approach? (Hint: There is a very good reason that this question is in this section!)

58. If $y = a \cdot x^m$, show that $\ln y = \ln a + m \ln x$. If $v = \ln y$, $u = \ln x$ and $b = \ln a$, show that $v = mu + b$. Explain why the graph of v as a function of u would be a straight line. This graph is called the **log-log plot** of y and x.

59. For the given data, compute $v = \ln y$ and $u = \ln x$, and plot points (u, v). Find constants m and b such that $v = mu + b$ and use the results of exercise 58 to find a constant a such that $y = a \cdot x^m$.

x	2.2	2.4	2.6	2.8	3.0	3.2
y	14.52	17.28	20.28	23.52	27.0	30.72

60. Repeat exercise 59 for the given data.

x	2.8	3.0	3.2	3.4	3.6	3.8
y	9.37	10.39	11.45	12.54	13.66	14.81

61. Construct a log-log plot (see exercise 58) of the U.S. population data in example 5.14. Compared to the semi-log plot of the data in Figure 0.75, does the log-log plot look linear? Based on this, are the population data modeled better by an exponential function or a polynomial (power) function?

62. Construct a semi-log plot of the data in exercise 59. Compared to the log-log plot already constructed, does this plot look linear? Based on this, are these data better modeled by an exponential or power function?

63. The concentration $[H^+]$ of free hydrogen ions in a chemical solution determines the solution's pH, as defined by $\text{pH} = -\log [H^+]$. Find $[H^+]$ if the pH equals (a) 7, (b) 8 and (c) 9. For each increase in pH of 1, by what factor does $[H^+]$ change?

64. Gastric juice is considered an acid, with a pH of about 2.5. Blood is considered alkaline, with a pH of about 7.5. Compare the concentrations of hydrogen ions in the two substances (see exercise 63).

65. The Richter magnitude M of an earthquake is defined in terms of the energy E in joules released by the earthquake, with $\log_{10} E = 4.4 + 1.5M$. Find the energy for earthquakes with magnitudes (a) 4, (b) 5 and (c) 6. For each increase in M of 1, by what factor does E change?

66. It puzzles some people who have not grown up around earthquakes that a magnitude 6 quake is considered much more severe than a magnitude 3 quake. Compare the amount of energy released in the two quakes. (See exercise 65.)

67. The decibel level of a noise is defined in terms of the intensity I of the noise, with $\text{dB} = 10 \log (I/I_0)$. Here, $I_0 = 10^{-12}\ \text{W/m}^2$ is the intensity of a barely audible sound. Compute the intensity levels of sounds with (a) $\text{dB} = 80$, (b) $\text{dB} = 90$ and (c) $\text{dB} = 100$. For each increase of 10 decibels, by what factor does I change?

68. At a basketball game, a courtside decibel meter shows crowd noises ranging from 60 dB to 110 dB. Compare the intensity level of the 110-dB crowd noise with that of the 60-dB noise. (See exercise 67.)

69. Use a graphing calculator to graph $y = xe^{-x}$, $y = xe^{-2x}$, $y = xe^{-3x}$ and so on. Estimate the location of the maximum for each. In general, state a rule for the location of the maximum of $y = xe^{-kx}$.

70. In golf, the task is to hit a golf ball into a small hole. If the ground near the hole is not flat, the golfer must judge how much the ball's path will curve. Suppose the golfer is at the point $(-13, 0)$, the hole is at the point $(0, 0)$ and the path of the ball is, for $-13 \le x \le 0$, $y = -1.672x + 72 \ln (1 + 0.02x)$. Show that the ball goes in the hole and estimate the point on the y-axis at which the golfer aimed.

Exercises 71–76 refer to the hyperbolic functions.

71. Show that the range of the hyperbolic cosine is $\cosh x \ge 1$ and the range of the hyperbolic sine is the entire real line.

72. Show that $\cosh^2 x - \sinh^2 x = 1$ for all x.

73. The Saint Louis Gateway Arch is both 630 feet wide and 630 feet tall. (Most people think that it looks taller than it is wide.) One model for the outline of the arch is $y = 757.7 - 127.7 \cosh \left(\frac{x}{127.7}\right)$ for $y \ge 0$. Use a graphing calculator to approximate the x- and y-intercepts and determine if the model has the correct horizontal and vertical measurements.

74. To model the outline of the Gateway Arch with a parabola, you can start with $y = -c(x + 315)(x - 315)$ for some constant c. Explain why this gives the correct x-intercepts. Determine the constant c that gives a y-intercept of 630. Graph this parabola and the hyperbolic cosine in exercise 73 on the same axes. Are the graphs nearly identical or very different?

75. Find all solutions of $\sinh (x^2 - 1) = 0$.

76. Find all solutions of $\cosh (3x + 2) = 0$.

77. On a standard piano, the A below middle C produces a sound wave with frequency 220 Hz (cycles per second). The frequency of the A one octave higher is 440 Hz. In general, doubling the frequency produces the same note an octave higher. Find an exponential formula for the frequency f as a function of the number of octaves x above the A below middle C.

78. There are 12 notes in an octave on a standard piano. Middle C is 3 notes above A (see exercise 77). If the notes are tuned equally, this means that middle C is a quarter-octave above A. Use $x = \frac{1}{4}$ in your formula from exercise 77 to estimate the frequency of middle C.

EXPLORATORY EXERCISES

1. Graph $y = x^2$ and $y = 2^x$ and approximate the two positive solutions of the equation $x^2 = 2^x$. Graph $y = x^3$ and $y = 3^x$, and approximate the two positive solutions of the equation $x^3 = 3^x$. Explain why $x = a$ will always be a solution of $x^a = a^x$, $a > 0$. What is different about the role of $x = 2$ as a solution of $x^2 = 2^x$ compared to the role of $x = 3$ as a solution of $x^3 = 3^x$? To determine the a-value at which the change occurs, graphically solve $x^a = a^x$ for $a = 2.1, 2.2, \ldots, 2.9$, and note that $a = 2.7$ and $a = 2.8$ behave differently. Continue to narrow down the interval of change by testing $a = 2.71, 2.72, \ldots, 2.79$. Then guess the exact value of a.

2. Graph $y = \ln x$ and describe the behavior near $x = 0$. Then graph $y = x \ln x$ and describe the behavior near $x = 0$. Repeat this for $y = x^2 \ln x$, $y = x^{1/2} \ln x$ and $y = x^a \ln x$ for a variety of positive constants a. Because the function "blows up" at $x = 0$, we say that $y = \ln x$ has a **singularity** at $x = 0$. The **order** of the singularity at $x = 0$ of a function $f(x)$ is the smallest value of a such that $y = x^a f(x)$ doesn't have a singularity at $x = 0$. Determine the order of the singularity at $x = 0$ for (a) $f(x) = \frac{1}{x}$, (b) $f(x) = \frac{1}{x^2}$ and (c) $f(x) = \frac{1}{x^3}$. The higher the order of the singularity, the "worse" the singularity is. Based on your work, how bad is the singularity of $y = \ln x$ at $x = 0$?